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On the bilinear exchange coupling in ferromagnetic multilayers

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Abstract

We investigate a mathematical model describing the bilinear interlayer exchange coupling (IEC) of ferromagnets through spacers. We propose an extension in the case of the Maxwell system of the results obtained in Hamdache K and Tilioua M (2004 *SIAM J. Appl. Math.* **64** 1077–97). The model couples the Landau–Lifshitz–Gilbert (LLG) equations with the Maxwell system. The Hoffmann interfacial boundary condition is considered to take into account bilinear IEC. The behavior of the electromagnetic field in the two cases of a thin and large nonmagnetic spacer is discussed. For example we obtain that the magnetic field in the nonmagnetic spacer vanishes in the case of a thin spacer. However the electric field depends explicitly on the initial data. Various other convergence results are also given.

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1. Introduction

In this work we deal with a mathematical model arising in the theory of the IEC for ferromagnets through spacers. For the theory and physical interests we may find, for example in [6, 10], and the references therein, many details and explanations.

The model considered in [5] is given by the LLG equation coupled with the magnetostatic approximation of the electromagnetic field. In this paper we shall extend in the case of the Maxwell system, the results obtained in [5].

Let us first describe the bilinear IEC model equations. We consider $B \subset \mathbb{R}^2$ a bounded and regular open set representing the cross section of the cylinder $\Omega = B \times (-1, 1)$ of $\mathbb{R}^3$. The generic point of $\mathbb{R}^3$ is denoted by $x = (\hat{x}, x_3)$ with $\hat{x} = (x_1, x_2) \in B$. We assume that a ferromagnetic material occupies the domains $\Omega_\varepsilon^- = B \times (-1, -\varepsilon)$ and $\Omega_\varepsilon^+ = B \times (\varepsilon, 1)$ separated by a spacer layer of nonmagnetic but electrically non-conductive material of

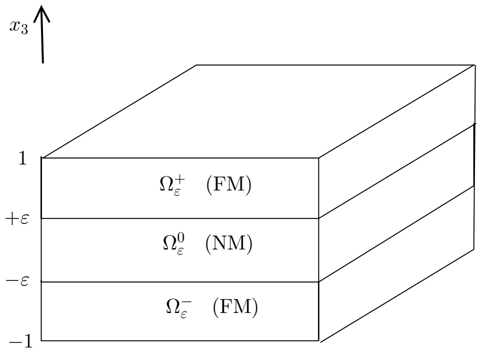

Figure 1. Schematic of a multilayer: Ω_ε^- and Ω_ε^+ are ferromagnetic material (FM). Ω_ε^0 is a non-magnetic material (NM).

thickness $2\varepsilon > 0$ occupying the domain $\Omega_\varepsilon^0 = B \times (-\varepsilon, \varepsilon)$ (see figure 1). In what follow, S^2 represents the unit sphere of $\mathbb{R}^3$, and we set $\Omega_\varepsilon = \Omega_\varepsilon^- \cup \Omega_\varepsilon^+$. The magnetization field of the ferromagnetic material which belongs to S^2 almost everywhere, is denoted by $M(t, x)$. Its evolution is governed by the LLG equations see [1, 3, 9] for example. We have

$$\begin{cases} \dfrac{1}{1+\alpha^2}(\partial_t M - \alpha M \times \partial_t M) = -M \times \mathcal{H}(M) \quad \text{in} \quad \mathbb{R}^+ \times \Omega_\varepsilon, \\ M(0, x) = M_0(x) \quad \text{in} \quad \Omega_\varepsilon, \qquad \partial_n M = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{1}$$

where the symbol $\times$ denotes the vector cross product in $\mathbb{R}^3$. The constant α represents the damping parameter. The effective magnetic field $\mathcal{H}$ depends on M and is given by

$$\mathcal{H}(M) = \operatorname{div}(A \operatorname{grad} M) + H, \tag{2}$$

where A is the exchange variable coefficient satisfying the usual ellipticity condition in Ω_ε and the second-order partial differential operator corresponds to the magnetic excitation associated with the exchange energy. The last term is the magnetic field. It satisfies with the electric field E the Maxwell equations in $\mathbb{R}^3_{-1} = \mathbb{R}^2 \times (-1, +\infty)$. For an absorbing medium, we have

$$\begin{cases} \eta\partial_t E + \sigma E - \operatorname{curl} H = 0 \\ \mu_0\partial_t H + \operatorname{curl} E = -\partial_t M \\ E(0, \cdot) = E_0(\cdot), \qquad H(0, \cdot) = H_0(\cdot), \end{cases} \tag{3}$$

where μ_0 is the vacuum permeability (for simplicity we assume that μ_0 is equal to 1 but its defined value is $4\pi \times 10^{-7}$ H m^{-1} in SI-units), $\eta(x)$ the permittivity function which takes two values $\eta_1 > 0$ in Ω_ε and $\eta_2 > 0$ in $\mathbb{R}^3_{-1}\backslash\overline{\Omega_\varepsilon}$. The conductivity function σ satisfies $\sigma(x) \geqslant \sigma_1 \geqslant 0$ in Ω_ε and $\sigma(x) = 0$ in $\mathbb{R}^3_{-1}\backslash\overline{\Omega_\varepsilon}$. These equations are supplemented with appropriate transmission boundary conditions, see section 3.1.

Note that for the sake of simplicity, the bulk uniaxial anisotropy field, generally taken linear in M, is not considered in (2) since it only induces more computations and has no mathematical influence on the results we obtain. Recall that its expression is $H_u = K_u(M - (M \cdot U)U)$ where U is the easy axis (fixed vector of $\mathbb{R}^3$) and K_u is a constant.

To take into account the bilinear IEC, equations (1) are supplemented with the so-called Hoffmann interlayer exchange coupling law [7, 8], which can be written as

$$M(\pm\varepsilon) \times \left(\mp A \frac{\partial M(\pm\varepsilon)}{\partial x_3} - J M(\mp\varepsilon) \right) = 0. \tag{4}$$

This condition couples the ferromagnetic layers of the domains Ω_ε^+ and Ω_ε^-. The constant J is the interlayer exchange coupling constant, which may depend on the thickness 2ε of the nonmagnetic spacer.

We define the energy

$$\mathcal{E}(t) = \int_{\Omega_\varepsilon} A|\operatorname{grad} M|^2\,\mathrm{d}x + \int_{\mathbb{R}^3_{-1}} (\eta|E|^2 + |H|^2)\,\mathrm{d}x - J\int_B M(-\varepsilon)M(\varepsilon)\,\mathrm{d}\hat{x}. \tag{5}$$

The following energy estimate holds.

Lemma 1.1. *If (M, E, H) is a solution of the problem (1)–(4) then, it satisfies, at least formally, the energy estimate*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) + 2\int_{\Omega_\varepsilon} \sigma|E|^2\,\mathrm{d}x + \frac{2\alpha}{1+\alpha^2}\int_{\Omega_\varepsilon} |\partial_t M|^2\,\mathrm{d}x = 0. \tag{6}$$

Proof. The techniques to obtain (6) are analogous to those used in [4, 5] and [11]. We rewrite the LLG equation (1) in the form

$$(\alpha\partial_t M - (1+\alpha^2)\mathcal{H}) = \alpha M \times (\alpha\partial_t M - (1+\alpha^2)\mathcal{H}) - (1+\alpha^2)\mathcal{H}. \tag{7}$$

Multiplying (7) by $(\alpha\partial_t M - (1+\alpha^2)\mathcal{H})$ and using the saturation constraint $|M|^2 = 1$ a.e. yields to

$$\frac{\alpha}{1+\alpha^2}|\partial_t M|^2 = \mathcal{H}\cdot\partial_t M. \tag{8}$$

Integrating (8) on Ω_ε, the right-hand side of (8) becomes

$$\int_{\Omega_\varepsilon} \mathcal{H}\cdot\partial_t M\,\mathrm{d}x = -\int_{\Omega_\varepsilon} A_{ij}\partial_i M\partial_j(\partial_t M)\,\mathrm{d}x + \int_{\Omega_\varepsilon} H\cdot\partial_t M\,\mathrm{d}x + \int_{\partial\Omega_\varepsilon} \partial_n M\partial_t M\,\mathrm{d}\sigma. \tag{9}$$

To convert the two last terms of the right-hand side of (9), we test the first equation of (3) by E, the second by H and make use of the Hoffmann boundary conditions (4). This allows us to get (6). □

The content of this paper is the following. In section 2, we give an existence result of global weak solutions to the coupled problem (1)–(3) with Hoffmann interfacial boundary condition (4).

Section 3 deals with the asymptotic behavior of solutions when the thickness parameter ε tends either to 0 (thin nonmagnetic spacer) or 1 (large nonmagnetic spacer). The main ingredients of our arguments are some *a priori* estimates on the solutions and passing to the limit in equations in the sense of distributions. We note that the limiting behavior of the magnetization field obtained in [5] remains valid, so we are interested only in the behavior of the electromagnetic field. We sometimes omit to write both the LLG equations and the Hoffmann interfacial boundary conditions coupling the magnetization at the interfaces. Let us specify the results obtained. We first introduce the changes of variables which transform the domains $\Omega_\varepsilon^\pm$ and Ω_ε^0 into domains which are independent of ε. We then give the rescaled system. In the case of a thin nonmagnetic spacer we assume that the interlayer exchange constant J is independent of ε. We pass to the limit in the Maxwell system, compatibility and transmission conditions. Depending on the infinite slab considered we show for example that in the slab $\mathbb{R}^+ \times S^0$ the magnetic field vanishes but the electric field is characterized by an explicit formula. Convergence results for other slabs are given. In the case of a large nonmagnetic spacer we show, for example by passing to the limit in the slab $\mathbb{R}^+ \times S^\pm$, that the magnetic and electric fields depend explicitly on the magnetization and initial electric fields, respectively. Results for other slabs are given.

Throughout, we use the following notations. $\mathbb{L}^2(\Omega) = (L^2(\Omega))^3$ and $\mathbb{H}^1(\Omega) = (H^1(\Omega))^3$ are the usual Hilbert spaces equipped with the norm $|\cdot|$ and $\|\cdot\|$, respectively. The same letter C denotes various positive constant which are all independent of ε. $\chi(\omega)$ will represent the characteristic function of ω. We make use of the following differential operators:

$$\begin{cases} \widehat{\operatorname{div}}\, g = \partial_x g + \partial_y g, & \widehat{\operatorname{curl}}\, g = \partial_x g_2 + \partial_y g_1 \\ \widehat{\operatorname{grad}}\, f = (\partial_x f, \partial_y f, 0), & \widehat{\mathbf{curl}}\, f = (\partial_y f, -\partial_x f, 0), \end{cases}$$

where f is a scalar function and g is a vectorial one. We set (u_1, u_2, u_3) to represent the canonical basis of $\mathbb{R}^3$.

2. Global existence of weak solutions

Before stating global existence result let us recall the definition of weak solutions to the system (1)–(4).

Definition 2.1. *Let $T > 0$ and M_0 in $\mathbb{H}^1(\Omega_\varepsilon)$, $|M_0| = 1$ a.e. in Ω_ε, (M, E, H) is called a weak solution to the system (1)–(4) if*

(1) $M \in \mathbb{H}^1((0, T) \times \Omega_\varepsilon)$, $|M| = 1$ a.e., and $M(0, \cdot) = M_0$ in the sense of traces.
(2) $E \in L^2\big(0, T; \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big)\big) \cap L^\infty\big(0, T; \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big)\big)$ and $H \in L^\infty(0, T; \mathbb{L}^2(\mathbb{R}^3_{-1}))$.
(3) M satisfies (1) and (E, H) satisfies (3) in the sense of distributions.
(4) For all $t \in [0, T]$, the energy estimate (6) holds.

We consider an initial distribution (M_0, E_0, H_0) satisfying

$$\begin{cases} M_0 \in \mathbb{H}^1(\Omega_\varepsilon), & |M_0(x)|^2 = 1 \text{ almost everywhere in } \Omega_\varepsilon, \\ E_0 \in \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big), & H_0 \in \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big) \end{cases} \tag{10}$$

then we have the following global existence result

Theorem 2.1. *There exists a global weak solution (M, E, H) to the problem (1)–(4) such that*

- *$|M(t, x)|^2 = 1$ almost everywhere in $\mathbb{R}^+ \times \Omega_\varepsilon$,*
- *$M \in L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega_\varepsilon))$, $\partial_t M \in L^2(\mathbb{R}^+; \mathbb{L}^2(\Omega_\varepsilon))$,*
- *$E \in L^2\big(\mathbb{R}^+; \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big)\big) \cap L^\infty\big(\mathbb{R}^+; \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big)\big)$,*
- *$H \in L^\infty\big(\mathbb{R}^+; \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big)\big)$.*

Moreover (E, H) satisfies the Maxwell system (3) and the energy estimate (6) holds.

Proof. We refer to Visintin [12] and Alouges–Soyeur [2] for a classical proof of the global existence of solutions. In our model the main difference is related to the coupling with the Maxwell system and the Hoffmann boundary condition satisfied by the magnetization. The proof follows almost exactly the proofs in [4, 5]. □

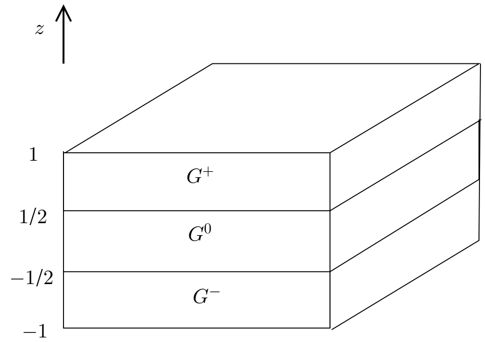

Figure 2. Fixed domain.

3. Convergences

The nonmagnetic spacer occupies the domain Ω_ε^0. It is convenient first to introduce the following change of variables:

$$\begin{cases} z = x_3, & \text{if} \quad x_3 \geqslant 1 \\ z = \dfrac{1}{2(1-\varepsilon)}(x_3 + 1 - 2\varepsilon) \in \left[\dfrac{1}{2}, 1\right], & \text{if} \quad \varepsilon \leqslant x_3 \leqslant 1 \\ z = \dfrac{1}{2\varepsilon} x_3 \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right], & \text{if} \quad -\varepsilon \leqslant x_3 \leqslant \varepsilon \\ z = \dfrac{1}{2(1-\varepsilon)}(x_3 - 1 + 2\varepsilon) \in \left[-1, -\dfrac{1}{2}\right], & \text{if} \quad -1 \leqslant x_3 \leqslant -\varepsilon. \end{cases} \tag{11}$$

We set

$$\begin{aligned} G^+ &= B \times \left(\tfrac{1}{2}, 1\right), & G^- &= B \times \left(-1, -\tfrac{1}{2}\right), \\ G^0 &= B \times \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), & G &= G^+ \cup G^-. \end{aligned} \tag{12}$$

A schematic representation of the fixed domain is provided in figure 2.

We define the new function m^ε by

$$m^\varepsilon(t, \hat{x}, z) = \begin{cases} M(t, \hat{x}, 2(1-\varepsilon)z + 2\varepsilon - 1) & \text{in} \quad \mathbb{R}^+ \times G^+ \\ M(t, \hat{x}, 2(1-\varepsilon)z - 2\varepsilon + 1) & \text{in} \quad \mathbb{R}^+ \times G^- \end{cases} \tag{13}$$

and the rescaled exchange coefficient a^ε by

$$a^\varepsilon(\hat{x}, z) = \begin{cases} A(\hat{x}, 2(1-\varepsilon)z + 2\varepsilon - 1) & \text{in} \quad G^+ \\ A(\hat{x}, 2(1-\varepsilon)z - 2\varepsilon + 1) & \text{in} \quad G^-. \end{cases} \tag{14}$$

We assume that $A \in L^\infty(B, C^0([-1, 1]))$. Note that a^ε satisfies the usual ellipticity condition.

We also introduce the slabs of $\mathbb{R}^3_{-1}$,

$$\begin{cases} S^- = \mathbb{R}^2 \times (-1, -1/2), \quad S^0 = \mathbb{R}^2 \times (-1/2, 1/2), \quad S^+ = \mathbb{R}^2 \times (1/2, 1), \\ S^\infty = \mathbb{R}^2 \times (1, \infty) \quad \text{and} \quad S^{+,\infty} = \mathbb{R}^2 \times (1/2, \infty), \quad S^{-,\infty} = \mathbb{R}^2 \times (-1, \infty) \end{cases} \tag{15}$$

and for $(t, \hat{x}, z) \in \mathbb{R}^+ \times \mathbb{R}^3_{-1}$ the function e^ε

$$e^\varepsilon(t,\hat{x},z) = \begin{cases} E(t,\hat{x},z), & \text{if} \quad z \geqslant 1 \\ E(t,\hat{x},2(1-\varepsilon)z+2\varepsilon-1), & \text{if} \quad \frac{1}{2} \leqslant z \leqslant 1, \\ E(t,\hat{x},2\varepsilon z), & \text{if} \quad -\frac{1}{2} \leqslant z \leqslant \frac{1}{2}, \\ E(t,\hat{x},2(1-\varepsilon)z-2\varepsilon+1), & \text{if} \quad -1 \leqslant z \leqslant -\frac{1}{2}. \end{cases} \tag{16}$$

Similarly we introduce the rescaled magnetic field h^ε.

3.1. Rescaled system

The rescaled LLG equations are

$$\partial_t m^\varepsilon - \alpha m^\varepsilon \times \partial_t m^\varepsilon = -(1+\alpha^2) m^\varepsilon \times \mathcal{H}^\varepsilon(m^\varepsilon), \tag{17}$$

where

$$\mathcal{H}^\varepsilon(m^\varepsilon) = \widehat{\mathrm{div}}(a^\varepsilon \widehat{\mathrm{grad}}\, m^\varepsilon) + \frac{1}{4(1-\varepsilon)^2}\partial_z(a^\varepsilon \partial_z m^\varepsilon) + h^\varepsilon.$$

The Hoffmann interfacial boundary condition (4) takes the form

$$m^\varepsilon(\pm 1/2) \times \left(\mp \frac{a^\varepsilon}{2(1-\varepsilon)} \partial_z m^\varepsilon(\pm 1/2) - J m^\varepsilon(\mp 1/2) \right) = 0. \tag{18}$$

In order to write the rescaled Maxwell system we introduce the function $\zeta^\varepsilon(z)$ defined by

$$\zeta^\varepsilon(z) = \begin{cases} 1, & \text{if} \quad z \geqslant 1 \\ \dfrac{1}{2(1-\varepsilon)}, & \text{if} \quad -1 \leqslant z \leqslant -\dfrac{1}{2} \text{ or } \dfrac{1}{2} < z < 1 \\ \dfrac{1}{2\varepsilon}, & \text{if} \quad -\dfrac{1}{2} \leqslant z \leqslant \dfrac{1}{2}. \end{cases} \tag{19}$$

In $\mathbb{R}^+ \times \mathbb{R}^3_{-1}$, the Maxwell system becomes by using the change of variables (11) and the function ζ^ε

$$\begin{cases} \partial_t(\eta^\varepsilon e^\varepsilon) - \widehat{\mathbf{curl}}(h^\varepsilon \cdot u_3) - (\widehat{\mathrm{curl}}\, h^\varepsilon) u_3 + \zeta^\varepsilon(z)\partial_z(h^\varepsilon \times u_3) + \sigma^\varepsilon e^\varepsilon = 0 \\ \partial_t(h^\varepsilon + \chi(G) m^\varepsilon) + \widehat{\mathbf{curl}}(e^\varepsilon \cdot u_3) + (\widehat{\mathrm{curl}}\, e^\varepsilon) u_3 - \zeta^\varepsilon(z)\partial_z(e^\varepsilon \times u_3) = 0 \\ e^\varepsilon(0) = e^\varepsilon_0; \qquad h^\varepsilon(0) = h^\varepsilon_0 \end{cases} \tag{20}$$

with the following compatibility conditions

$$\begin{cases} \partial_t \widehat{\mathrm{div}}(\eta^\varepsilon \widehat{e^\varepsilon}) + \zeta^\varepsilon(z)\partial_t\partial_z(\eta^\varepsilon e^\varepsilon \cdot u_3) + \widehat{\mathrm{div}}(\sigma^\varepsilon \widehat{e^\varepsilon}) + \zeta^\varepsilon(z)\partial_z(\sigma^\varepsilon e^\varepsilon \cdot u_3) = 0 \\ \widehat{\mathrm{div}}(\widehat{h^\varepsilon} + \chi(G)\widehat{m^\varepsilon}) + \partial_z(h^\varepsilon \cdot u_3 + \chi(G) m^\varepsilon \cdot u_3) = 0. \end{cases} \tag{21}$$

To find a solution to a specific electromagnetic problem, the Maxwell equations should be supplemented with a transmission boundary conditions associated with a considered domain. Formally, for two mediums i and j, corresponding to the disjoint domains Ω_i and Ω_j such that $\partial\Omega_i \cap \partial\Omega_j = \Gamma_{ij} \neq \emptyset$ we have $n \times (E_i - E_j) = 0$; $n \times (H_i - H_j) = J_{\mathrm{surf}}$; $n \cdot (D_i - D_j) = \rho_{\mathrm{surf}}$ and $n \cdot (B_i - B_j) = 0$, where n is the normal on Γ_{ij} pointing from Ω_j to Ω_i. J_{surf} is the surface electric density and ρ_{surf} is the surface charge density. In our case, we write for the electric field

$$\begin{cases} \dfrac{1}{2\varepsilon} e^\varepsilon(-1/2^+) \times u_3 = \dfrac{1}{2(1-\varepsilon)} e^\varepsilon(-1/2^-) \times u_3 \\ \dfrac{1}{2(1-\varepsilon)} e^\varepsilon(1/2^+) \times u_3 = \dfrac{1}{2\varepsilon} e^\varepsilon(1/2^-) \times u_3 \\ e^\varepsilon(1^+) \times u_3 = \dfrac{1}{2(1-\varepsilon)} e^\varepsilon(1^-) \times u_3 \end{cases} \tag{22}$$

and

$$\begin{cases} \dfrac{1}{2\varepsilon}e^{\varepsilon}(-1/2^{+})\cdot u_3 = \dfrac{1}{2(1-\varepsilon)}e^{\varepsilon}(-1/2^{-})\cdot u_3 \\ \dfrac{1}{2(1-\varepsilon)}e^{\varepsilon}(1/2^{+})\cdot u_3 = \dfrac{1}{2\varepsilon}e^{\varepsilon}(1/2^{-})\cdot u_3 \\ e^{\varepsilon}(1^{+})\cdot u_3 = \dfrac{1}{2(1-\varepsilon)}e^{\varepsilon}(1^{-})\cdot u_3. \end{cases} \tag{23}$$

For the magnetic field, we have

$$\begin{cases} \dfrac{1}{2\varepsilon}h^{\varepsilon}(-1/2^{+})\times u_3 = \dfrac{1}{2(1-\varepsilon)}h^{\varepsilon}(-1/2^{-})\times u_3 \\ \dfrac{1}{2(1-\varepsilon)}h^{\varepsilon}(1/2^{+})\times u_3 = \dfrac{1}{2\varepsilon}h^{\varepsilon}(1/2^{-})\times u_3 \\ h^{\varepsilon}(1^{+})\times u_3 = \dfrac{1}{2(1-\varepsilon)}h^{\varepsilon}(1^{-})\times u_3 \end{cases} \tag{24}$$

and

$$\begin{cases} \dfrac{1}{2\varepsilon}h^{\varepsilon}(-1/2^{+})\cdot u_3 = \dfrac{\varepsilon}{(1-\varepsilon)}(h^{\varepsilon}(-1/2^{-})\cdot u_3 + \chi(G)m^{\varepsilon}(-1/2^{-})\cdot u_3) \\ h^{\varepsilon}(1/2^{-})\cdot u_3 = \dfrac{\varepsilon}{(1-\varepsilon)}(h^{\varepsilon}(1/2^{+})\cdot u_3 + \chi(G)m^{\varepsilon}(1/2^{+})\cdot u_3) \\ h^{\varepsilon}(1^{+})\cdot u_3 = \dfrac{1}{2(1-\varepsilon)}(h^{\varepsilon}(1^{-})\cdot u_3 + \chi(G)m^{\varepsilon}(1^{-})\cdot u_3). \end{cases} \tag{25}$$

The energy estimate (6) becomes

$$\mathcal{E}^{\varepsilon}(t) + \frac{2\alpha}{1+\alpha^2}\int_0^t\int_G |\partial_t m^{\varepsilon}(s)|^2\,\mathrm{d}x\,\mathrm{d}s \leqslant \mathcal{E}^{\varepsilon}(0), \tag{26}$$

where the energy $\mathcal{E}^{\varepsilon}(t)$ is expressed by

$$\begin{aligned} \mathcal{E}^{\varepsilon}(t) = \int_G a^{\varepsilon}|\widehat{\mathrm{grad}}\, m^{\varepsilon}|^2\,\mathrm{d}x + \frac{1}{4(1-\varepsilon)^2}\int_G a^{\varepsilon}|\partial_z m^{\varepsilon}|^2\,\mathrm{d}x \\ - \frac{J}{2(1-\varepsilon)}\int_B m^{\varepsilon}\left(-\frac{1}{2}\right)\cdot m^{\varepsilon}\left(\frac{1}{2}\right)\mathrm{d}\hat{x} + \int_{\mathbb{R}^3_{-1}} (\eta^{\varepsilon}|e^{\varepsilon}|^2 + |h^{\varepsilon}|^2)\,\mathrm{d}x. \end{aligned} \tag{27}$$

Our goal in the following subsections is to identify the limiting problem and characterize the limiting electromagnetic field. We will assume the following strong convergences of the rescaled electric conductivity and permittivity $\sigma^{\varepsilon} \to \sigma$, $\eta^{\varepsilon} \to \eta$.

3.2. Asymptotic behavior for a thin nonmagnetic spacer

We discuss the behavior of the solutions $(m^{\varepsilon}, e^{\varepsilon}, h^{\varepsilon})$ when $\varepsilon \to 0$. In view of the energy estimate (26), to get uniform bounds on the solutions we consider an initial data m_0^{ε} which is independent of z such that $\left|m_0^{\varepsilon}\right|_{\mathbb{H}^1(B)} \leqslant C$ and $\left|m_0^{\varepsilon}(\hat{x})\right|^2 = 1$ a.e. We have the following result.

Theorem 3.1. Let $(m^{\varepsilon}, e^{\varepsilon}, h^{\varepsilon})$ be a global solution to (17)–(20). Let (m, e, h) be the weak-$\star$ limit of a subsequence $(m^{\varepsilon}, e^{\varepsilon}, h^{\varepsilon})$ in $L^{\infty}(\mathbb{R}^+, \mathbb{H}^1(G)) \times (L^{\infty}(\mathbb{R}^+, \mathbb{L}^2(\mathbb{R}^3_{-1})))^2$. Then (m, e, h) satisfies in $\mathbb{R}^+ \times G$ the following equation:

$$\begin{cases} \partial_t m - \alpha m \times \partial_t m = -(1+\alpha^2) m \times \mathcal{H}(m) \quad \text{in} \quad \mathbb{R}^+ \times (G^- \cup G^+), \\ m(0,x) = m_0(x),\ \partial_n m = 0 \quad \text{in} \quad \partial\Omega\backslash\{z = \pm 1/2\} \\ m(\pm 1/2) \times (\mp a\partial_z m(\pm 1/2) - 2Jm(\mp 1/2)) = 0 \quad \text{in} \quad B \\ \mathcal{H}(m) = \widehat{\mathrm{div}}(a\,\widehat{\mathrm{grad}}\, m) + \frac{1}{4}\partial_z(a\partial_z m) + h, \end{cases} \tag{28}$$

where h is the limiting magnetic field.

Proof. The behavior of the magnetization field obtained in [5] remains valid. We need in fact to characterize the magnetic field h appearing in the effective field $\mathcal{H}$. This will be done in several steps. The energy estimate (26) allows us to deduce the following convergences.

Lemma 3.1. *The solutions* $(e^\varepsilon, h^\varepsilon)$ *satisfy (up to extract subsequences) the following convergences:*

$$(e^\varepsilon, h^\varepsilon) \rightharpoonup (e, h) \text{ weakly in } L^2\big(\mathbb{R}^+; \mathbb{L}^2\big(\mathbb{R}^3_{-1}\big)\big). \tag{29}$$

Now we pass to the limit in the transmission conditions (22)–(25) we obtain

$$\begin{cases} e^0(-1/2^+) \times u_3 = e^0(-1/2^+) \cdot u_3 = 0 \\ e^0(1/2^-) \times u_3 = e^0(1/2^-) \cdot u_3 = 0 \\ h^0(-1/2^+) \times u_3 = h^0(-1/2^+) \cdot u_3 = 0 \\ h^0(1/2^-) \times u_3 = h^0(1/2^-) \cdot u_3 = 0 \end{cases} \tag{30}$$

and

$$\begin{cases} 2e^\infty(1^+) \times u_3 = e^+(1^-) \times u_3 \\ 2e^\infty(1^+) \cdot u_3 = e^+(1^-) \cdot u_3 \\ 2h^\infty(1^+) \times u_3 = h^+(1^-) \times u_3 \\ 2h^\infty(1^+) \cdot u_3 = h^+(1^-) \cdot u_3 + \chi(G) m^+(1^-) \cdot u_3. \end{cases} \tag{31}$$

We begin by the slab S^0. We multiply equation (20), for $\zeta^\varepsilon(z) = 1/2\varepsilon$, by ε. We pass to the limit when $\varepsilon \to 0$ in the sense of distributions by using the weak convergences results (29). We obtain

$$e^0(0) = e^0_0 \quad \text{and} \quad h^0(0) = h^0_0 \quad \text{in} \quad S^0 \tag{32}$$

$$\partial_z(h^0 \times u_3) = 0 \quad \text{and} \quad \partial_z(e^0 \times u_3) = 0 \quad \text{in} \quad \mathbb{R}^+ \times S^0. \tag{33}$$

Now we pass to the limit in the free divergence condition (21) to obtain

$$\partial_t \partial_z(\eta e^0 \cdot u_3) + \partial_z(\sigma e^0 \cdot u_3) = 0 \quad \text{and} \quad \partial_z(h^0 \cdot u_3) = 0 \quad \text{in} \quad \mathbb{R}^+ \times S^0. \tag{34}$$

The equations (33) and (34) imply that

$$e^0 \cdot u_3 = e^0_0 \cdot u_3 \exp\left(-\frac{\sigma}{\eta} t\right). \tag{35}$$

We also conclude that the magnetic field h^0 is independent of z, i.e.,

$$h^0 = h^0(t, \hat{x}) \text{ almost everywhere } (t, \hat{x}). \tag{36}$$

The third equality of equation (30) allows us to conclude that the magnetic field h vanishes in the slab S^0 since it is independent of z. On the other hand, the second equality of equation (33) and the second equality of equation (30) give $e^0 = e^0 \cdot u_3 u_3$. From the characterization (35) we deduce that

$$e^0 = e^0_0 \cdot u_3 \exp\left(-\frac{\sigma}{\rho} t\right) u_3. \tag{37}$$

We proved the following result.

Proposition 3.1. *In* $\mathbb{R}^+ \times S^0$, *the magnetic field* h^0 *vanishes and the electric field* e^0 *is characterized by (37).*

Now we pass to the limit in the slabs $S^{\pm}$. We use the bounds on the solutions $(m^{\varepsilon}, e^{\varepsilon}, h^{\varepsilon})$. We get

$$\begin{cases} \partial_t(\eta e^{\pm}) - \widehat{\mathbf{curl}}(h^{\pm} \cdot u_3) - (\widehat{\mathrm{curl}}\, h^{\pm})u_3 + \frac{1}{2}\partial_z(h^{\pm} \times u_3) + \sigma e^{\pm} = 0 \\ \partial_t(h^{\pm} + \chi(G^{\pm})m^{\pm}) + \widehat{\mathbf{curl}}(e^{\pm} \cdot u_3) + (\widehat{\mathrm{curl}}\, e^{\pm})u_3 - \frac{1}{2}\partial_z(e^{\pm} \times u_3) = 0 \\ e^{\pm}(0) = e_0^{\pm}; \qquad h^{\pm}(0) = h_0^{\pm}; \qquad e^{-} \times u_3 = 0 \quad \text{at} \quad z = -1. \end{cases} \tag{38}$$

On the other hand, we pass to the limit in the free divergence conditions (21), we get

$$\begin{cases} \partial_t \widehat{\mathrm{div}}(\eta \widehat{e}^{\pm}) + \frac{1}{2}\partial_t\partial_z(\eta e^{\pm} \cdot u_3) + \widehat{\mathrm{div}}(\sigma \widehat{e}^{\pm}) + \frac{1}{2}\partial_z(\sigma e^{\pm} \cdot u_3) = 0 \\ \widehat{\mathrm{div}}(\widehat{h}^{\pm} + \chi(G)\widehat{m}^{\pm}) + \frac{1}{2}\partial_z(h^{\pm} \cdot u_3 + \chi(G^{\pm})m \pm \cdot u_3) = 0 \end{cases} \tag{39}$$

in $\mathbb{R}^{+} \times S^{\pm}$.

We proved the following result.

Proposition 3.2. *In $\mathbb{R}^{+} \times S^{\pm}$, the couple $(e^{\pm}, h^{\pm})$ is characterized by equations (38) and (39).*

Proposition 3.3. *In $\mathbb{R}^{+} \times S^{\infty}$, the limiting electromagnetic field satisfies the usual Maxwell system together with the corresponding compatibility conditions*

$$\begin{cases} \partial_t(\eta e^{\infty}) - \mathrm{curl}\, h^{\infty} + \sigma e^{\infty} = 0 \\ \partial_t(h^{\infty}) + \mathrm{curl}\, e^{\infty} = 0 \\ e^{\infty}(0) = e_0^{\infty}, \qquad h^{\infty}(0) = h_0^{\infty} \end{cases} \tag{40}$$

with

$$\begin{cases} \partial_t \mathrm{div}(\eta e^{\infty}) + \mathrm{div}(\sigma e^{\infty}) = 0 \\ \partial_t \mathrm{div}(h^{\infty}) = 0. \end{cases} \tag{41}$$

The characterization of the limiting electromagnetic field (e, h) is now complete (propositions 3.1, 3.2 and 3.3), and so finishes up the proof of theorem 3.1. □

3.3. Asymptotic behavior for large nonmagnetic spacer

We now focus on the behavior of the solutions $(m^{\varepsilon}, e^{\varepsilon}, h^{\varepsilon})$ when $\varepsilon \to 1$. We will pass to the limit in the Maxwell system (20) and the transmission conditions (21). We apply the same reasoning as above. In order to get uniform bounds on the solutions we consider an initial data $m_0^{\varepsilon} \in \mathbb{H}^1(B)$ such that $|m_0^{\varepsilon}(\hat{x})|^2 = 1$ and assume that the interlayer exchange coefficient is such that $J = j(1-\varepsilon)$, $j > 0$. We have the following result.

Theorem 3.2. *Let $m^{\pm}$ be the weak-⋆ in $L^{\infty}(\mathbb{R}^{+}; \mathbb{H}^1(G^{\pm}))$ of a subsequence of $m^{\varepsilon}_{|G^{\pm}}$. Then the couple (m^{+}, m^{-}) is independent of z and satisfies the saturation constraint $|m^{\pm}(t, \hat{x})|^2 = 1$ a.e. Moreover the couple (m^{+}, m^{-}) satisfies in $\mathbb{R}^{+} \times B$ the following LLG equations:*

$$\begin{cases} \partial_t m^{\pm} - \alpha m^{\pm} \times \partial_t m^{\pm} = -(1+\alpha^2) m^{\pm} \times \mathcal{H}^{\pm}(m^{\pm}) \\ m^{\pm}(0, \hat{x}) = m_0(\hat{x}), \qquad \partial_n m^{\pm} = 0 \quad \textit{on} \quad \partial B, \end{cases} \tag{42}$$

where the effective magnetic field $\mathcal{H}^{\pm}(m^{\pm})$ is given by

$$\mathcal{H}^{\pm}(m^{\pm}) = \widehat{\mathrm{div}}(a^{\pm}\widehat{grad}\, m^{\pm}) + h + j m^{\pm} \tag{43}$$

and h is the limiting magnetic field.

Proof. We refer to [5] for the limiting behavior of magnetization. We intend to characterize the limiting magnetic field h. This will be done in several steps. From the energy estimate (26) we get the following convergences.

Lemma 3.2. *The electromagnetic field satisfies*

$$\begin{cases} (e^\varepsilon, h^\varepsilon) \rightharpoonup (e, h) \text{ weakly-}\star \quad \text{in} \quad L^\infty(\mathbb{R}^+; \mathbb{L}^2(\mathbb{R}^3_{-1}) \\ e^\varepsilon \rightharpoonup e \text{ weakly in } L^2(\mathbb{R}^+, \mathbb{L}^2(\Omega)). \end{cases} \tag{44}$$

Now we characterize the limit electromagnetic field in each infinite slab. First, we pass to the limit ($\varepsilon \to 1$) in the transmission conditions (22)–(25). For the electric field we have

$$\begin{cases} e^-(-1/2^-) \times u_3 = e^-(1/2^-) \cdot u_3 = 0 \\ e^+(1/2^+) \times u_3 = e^+(1/2^+) \cdot u_3 = 0 \\ e^+(1^-) \times u_3 = e^+(1^-) \cdot u_3 = 0. \end{cases} \tag{45}$$

For the magnetic field we have

$$h^-(-1/2^-) \times u_3 = h^+(1/2^+) \times u_3 = h^+(1^-) \times u_3 = 0 \tag{46}$$

and

$$\begin{cases} h^-(-1/2^-) \cdot u_3 + \chi(G) m^-(-1/2^-) \cdot u_3 = 0 \\ h^+(1/2^+) \cdot u_3 + \chi(G) m^+(1/2^+) \cdot u_3 = 0 \\ h^+(1^-) \cdot u_3 + \chi(G) m^+(1^-) \cdot u_3 = 0. \end{cases} \tag{47}$$

Proposition 3.4. *The couple (e^0, h^0) satisfies in $\mathbb{R}^+ \times S^0$ the following system:*

$$\begin{cases} \partial_t(\eta e^0) - \widehat{\mathbf{curl}}(h^0 \cdot u_3) - (\widehat{\operatorname{curl}}\, h^0) u_3 + \frac{1}{2}\partial_z(h^0 \times u_3) + \sigma e^0 = 0 \\ \partial_t(h^0) + \widehat{\mathbf{curl}}(e^0 \cdot u_3) + (\widehat{\operatorname{curl}}\, e^0) u_3 - \frac{1}{2}\partial_z(e^0 \times u_3) = 0 \\ e^0(0) = e^0_0, \qquad h^0(0) = h^0_0 \end{cases} \tag{48}$$

together with the divergence free condition

$$\begin{cases} \widehat{\operatorname{div}}(\eta \widehat{e}^0) + \frac{1}{2}\partial_z(\eta e^0 \cdot u_3) = 0 \quad \text{in} \quad \mathbb{R}^+ \times S^0 \\ \widehat{\operatorname{div}}(\widehat{h}^0) + \frac{1}{2}\partial_z(h^0 \cdot u_3) = 0 \quad \text{in} \quad \mathbb{R}^+ \times S^0. \end{cases} \tag{49}$$

Proof. We pass to the limit in the sense of distributions when $\varepsilon \to 1$ in equations (20) and (21) by using the convergences (44). □

Consider now equations (20) in $\mathbb{R}^+ \times S^\pm$. We multiply them by $2(1-\varepsilon)$ and pass to the limit, in the sense of distributions. We obtain

$$\partial_z(h^\pm \times u_3) = 0 \quad \text{and} \quad \partial_z(e^\pm \times u_3) = 0. \tag{50}$$

Passing to the limit in the free divergence conditions we get

$$\begin{aligned} &\partial_t(\partial_z(\eta e^\pm \cdot u_3)) + \partial_z(\sigma e^\pm \cdot u_3) = 0 \\ &\partial_z(h^\pm \cdot u_3 + \chi(G) m^\pm \cdot u_3) = 0. \end{aligned} \tag{51}$$

We integrate equalities (51) by using (45) and (46); we get the following characterization of the limiting electromagnetic field in the ferromagnetic media.

Proposition 3.5. *In* $\mathbb{R}^+ \times S^{\pm}$*, the couple* $(e^{\pm}, h^{\pm})$ *is characterized by*

$$e^{\pm} = \exp\left(-t\frac{\sigma}{\eta}\right)(e_0^{\pm} \cdot u_3)u_3 \tag{52}$$

and

$$h^{\pm} = -\chi(\omega)\chi^{\pm}(z)(m^{\pm} \cdot u_3)u_3, \tag{53}$$

where $\chi^{\pm}(z)$ *is the characteristic function of* $(1/2, 1)$ *and* $(-1, -1/2)$.

Finally, in the slab S^{∞}, we get by taking the limit in (20) and (21).

Proposition 3.6. *In* $\mathbb{R}^+ \times S^{\infty}$*, the couple* (e^{∞}, h^{∞}) *satisfies the classical Maxwell system.*

The characterization of the limiting electromagnetic field (e, h) is now complete (propositions 3.4–3.6), and so finishes up the proof of theorem 3.2. □

4. Concluding remarks

We have explored the limiting behavior of the electromagnetic field in ferromagnetic multilayers consisting of magnetic layers separated by nonmagnetic ones. We have devoted this study to the coupling with perfect interfaces. It would be interesting to consider the effects of nonmagnetic impurities and interface roughness on the interlayer coupling between magnetic layers. Another direction for future research is to extend the results when the contribution of biquadratic coupling to interlayer exchange coupling is considered.

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